

Titouan Carette: Quacs, Université Paris Saclay

Joint work with Pablo Arrighi, Yohann D'Anello, Marc De Visme, Emmanuel Jeandel, Etienne Moutot, Simon Perdrix and Renaud Vilmart.

ZX-calculus

A Swiss army katana for quantum computing

Séminaire CANA

Chapter 1:

Graphical Languages

From diagrams to matrices

$$n \left\{ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{D} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right\} m \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad m \left\{ \overbrace{\left(\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \vdots \\ \text{---} \end{array} \right)}^n \right\}$$

$D : n \rightarrow m$ $\llbracket D \rrbracket \in \mathcal{M}_{2^m \times 2^n}(\mathbb{C})$

Compositionality

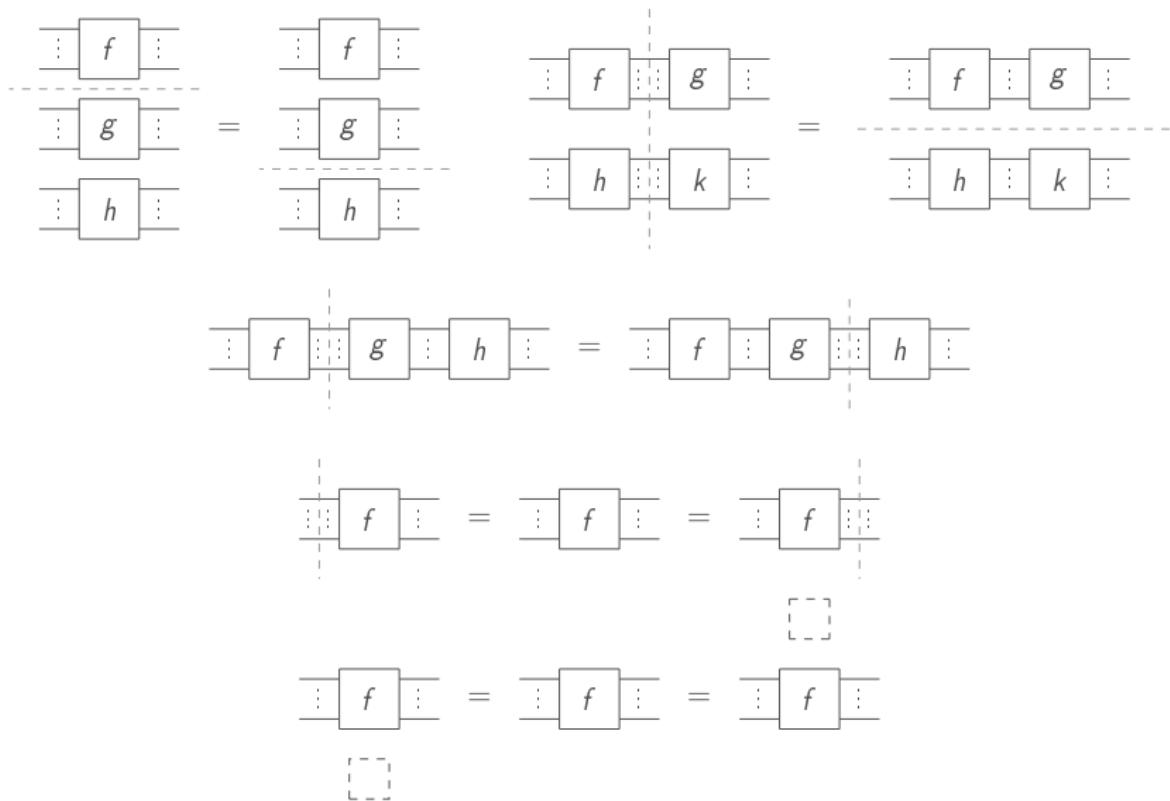
$$\left[\begin{array}{c|c} \vdots & f \\ \hline \vdots & g \end{array} \right] = \left[\begin{array}{c|c} \vdots & f \\ \hline \vdots & \end{array} \right] \otimes \left[\begin{array}{c|c} \vdots & g \\ \hline \vdots & \end{array} \right]$$
$$f : a \rightarrow b \qquad \qquad \qquad g : c \rightarrow d$$
$$f \otimes g : a + c \rightarrow b + d$$

$$\left[\begin{array}{c|c|c} \vdots & f & g \\ \hline \vdots & \vdots & \vdots \end{array} \right] = \left[\begin{array}{c|c} \vdots & g \\ \hline \vdots & \end{array} \right] \circ \left[\begin{array}{c|c} \vdots & f \\ \hline \vdots & \end{array} \right]$$
$$g \circ f : a \rightarrow d \qquad \qquad \qquad f : a \rightarrow b \qquad \qquad \qquad g : b \rightarrow c$$

$$\left[\begin{array}{c|c} \vdots & \vdots \\ \hline \vdots & \vdots \end{array} \right] = I_n$$
$$I_n : n \rightarrow n$$

$$\left[\begin{array}{c|c} \vdots & \vdots \\ \hline \vdots & \vdots \end{array} \right] = 1$$
$$I_0 : 0 \rightarrow 0$$

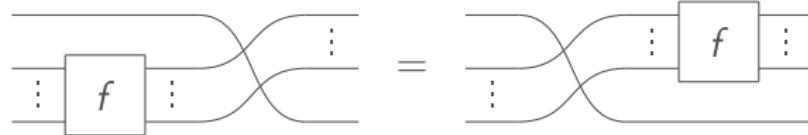
Picturing tautologies



Swaps



$$\sigma_2 : 2 \rightarrow 2$$



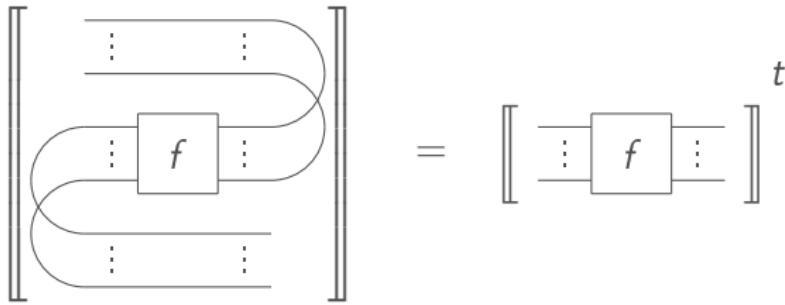
Cups and caps



$$\eta : 0 \rightarrow 2$$

$$\epsilon : 2 \rightarrow 0$$

$$\infty = (\quad S = - = \bar{2} \quad) = \infty$$



Chapter 2:

Vanilla ZX-Calculus

The generators of ZX-calculus

⊕ The wires:

$$[\![\text{---}]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |x\rangle \mapsto |x\rangle \quad [\![\text{---} \curvearrowright \text{---}]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = |xy\rangle \mapsto |yx\rangle$$

$$[\![\text{---} \text{---}]\!] = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |00\rangle + |11\rangle \quad [\![\text{---} \text{---}]\!] = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = \langle 00| + \langle 11|$$

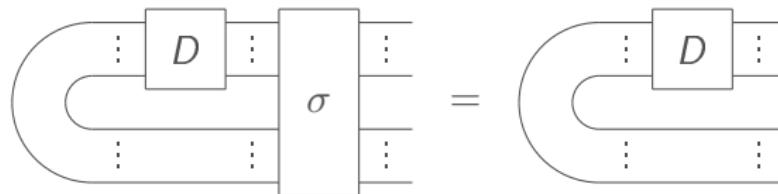
⊕ The generators:

$$[\![\text{---} \square \text{---}]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad [\![\text{---} \text{---} \text{---} \alpha]\!] = |\vec{0}\rangle\langle\vec{0}| + e^{i\alpha}|\vec{1}\rangle\langle\vec{1}|$$

$$[\![\text{---} \text{---} \text{---} \alpha]\!] = [\![\text{---} \square \text{---}]\!]^{\otimes m} \circ [\![\text{---} \text{---} \text{---} \alpha]\!] \circ [\![\text{---} \square \text{---}]\!]^{\otimes n}$$

Flexsymmetry

A diagram $D : n \rightarrow m$ is said **flexsymmetric** if for all permutation $\sigma \in \mathfrak{S}_{n+m}$:



The permutation σ is obtained by composing swaps.

The equations of ZX-calculus

$$\begin{array}{c} \text{Diagram: } \text{Two green nodes labeled } \alpha \text{ and } \beta \text{ connected by a vertical line. The top node has two outgoing wires, the bottom node has two incoming wires. They are connected by a horizontal wire.} \\ = \\ \text{Diagram: } \text{A green node labeled } \alpha + \beta \text{ with two outgoing wires.} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{Two red nodes labeled } \alpha \text{ and } \beta \text{ connected by a vertical line. The top node has two outgoing wires, the bottom node has two incoming wires. They are connected by a horizontal wire.} \\ = \\ \text{Diagram: } \text{A red node labeled } \alpha + \beta \text{ with two outgoing wires.} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{Two green nodes and two red nodes arranged in a 2x2 grid. The top-left is green, top-right is red, bottom-left is green, bottom-right is red. Wires cross between them.} \\ = \\ \text{Diagram: } \text{Two red nodes connected by a horizontal wire, followed by two green nodes connected by a horizontal wire.} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{Two red nodes connected by a horizontal wire.} \\ = \\ \text{Diagram: } \text{A red node connected to a green node.} \end{array}$$

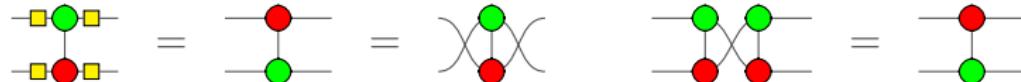
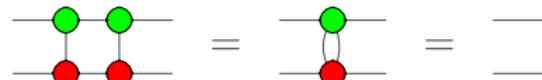
$$\begin{array}{c} \text{Diagram: } \text{A green node labeled } \alpha \text{ with four yellow squares at its corners.} \\ = \\ \text{Diagram: } \text{A red node labeled } \alpha \text{ with two outgoing wires.} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{A green node connected to a red node.} \\ = \\ \text{Diagram: } \text{An empty dashed box.} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{A green node labeled } \alpha \text{ followed by a yellow square, followed by a green node labeled } \alpha_2. \\ = \\ \text{Diagram: } \text{A red node labeled } \beta_1 \text{ followed by a green node labeled } \beta_2 \text{ followed by a red node labeled } \beta_3. \end{array}$$

A tell of CNOTs

$$\left[\begin{array}{c} \text{CNOT} \\ \text{CNOT} \end{array} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |xy\rangle \mapsto |x\rangle|x\oplus y\rangle$$



Chapter 3:

Discard ZX-Calculus

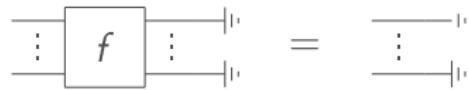
Mixed states

We extend our semantic to consider probabilistic mixtures of qubits.

- ⊕ Density matrix ρ of the form $\rho = rr^\dagger$.
- ⊕ $|x\rangle \mapsto |x\rangle\langle x|$.
- ⊕ Given $D : n \rightarrow m$ we have $\llbracket D \rrbracket^\pm : \mathcal{M}_{2^n \times 2^n}(\mathbb{C}) \rightarrow \mathcal{M}_{2^m \times 2^m}(\mathbb{C})$ completely positive.
- ⊕ Given $V : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^m}$ we have $\rho \mapsto V\rho V^\dagger$.

Discard map

$$[\neg \vdash] = \rho \mapsto Tr(\rho)$$



Equations of discard ZX-calculus

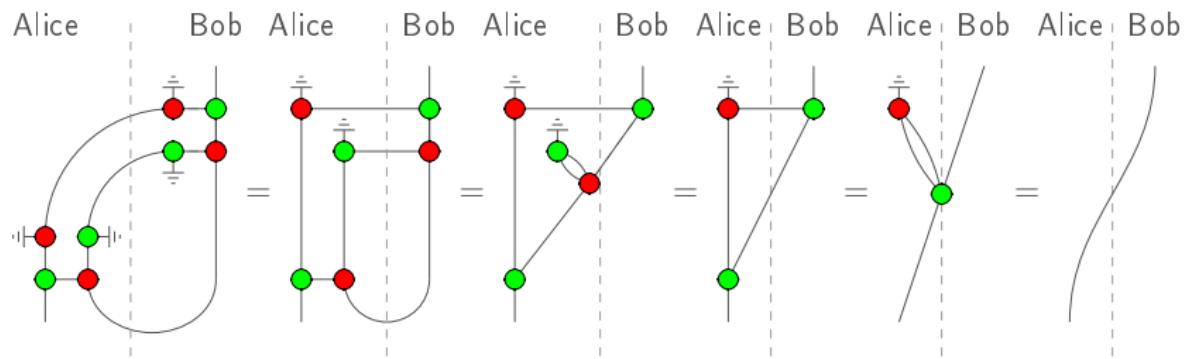
$$\text{---} \circlearrowleft \text{---} = \text{---}$$

$$\text{---} \square \text{---} = \text{---}$$

$$\begin{array}{c} \text{---} \circlearrowleft \text{---} \\ \text{---} \square \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$\bullet \text{---} = [\quad]$$

Quantum teleportation



Chapter 4:

Scalable ZX-Calculus

Register types

We can now gather qubits into registers using the operator $[\cdot]$.

- ⊕ $[n]$ is the type of a register of size n .
- ⊕ $[0] = 0$.
- ⊕ $[1] + [1] \neq [2]$, however those two types are isomorphic.
- ⊕ Given a gate $G : n \rightarrow m$ we can form a scaled gate $S_k G : [k]^{\otimes n} \rightarrow [k]^{\otimes m}$.

Divide and gather

$$[n+1] \xrightarrow{\quad} \begin{bmatrix} 1 \\ n \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ n \end{bmatrix} \xrightarrow{\quad} [n+1]$$

$$\text{---} = \text{---}$$

$$\text{---} = \text{---}$$

$$\vdots \quad S_{k+1} G \quad \vdots \quad \text{---} = \quad \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \quad \begin{array}{c} G \\ \text{---} \\ \vdots \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \quad \begin{array}{c} S_k G \\ \text{---} \\ \vdots \\ \text{---} \end{array}$$

Matrices

$$\begin{array}{c} A \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} A \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \left[\begin{array}{c} A \\ \text{---} \end{array} \right] : |x\rangle \mapsto |Ax\rangle$$

$$\begin{array}{c} A \\ \text{---} \end{array} \bullet \text{---} = \text{---}$$

$$\bullet \begin{array}{c} A \\ \text{---} \end{array} = \bullet \text{---}$$

$$\begin{array}{c} A \\ \text{---} \end{array} \bullet \text{---} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} A \\ \text{---} \end{array} \quad \begin{array}{c} A \\ \text{---} \end{array}$$

$$\begin{array}{c} A \\ \text{---} \end{array} \quad \begin{array}{c} A \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} A \\ \text{---} \end{array}$$

$$\bullet \begin{array}{c} A \\ \text{---} \end{array} = \bullet Ax$$

$$\begin{array}{c} A \\ \text{---} \end{array} \quad \square = \quad \square \begin{array}{c} A^t \\ \text{---} \end{array}$$

$$\begin{array}{c} A \quad B \\ \text{---} \end{array} = \begin{array}{c} \text{---} \end{array} BA$$

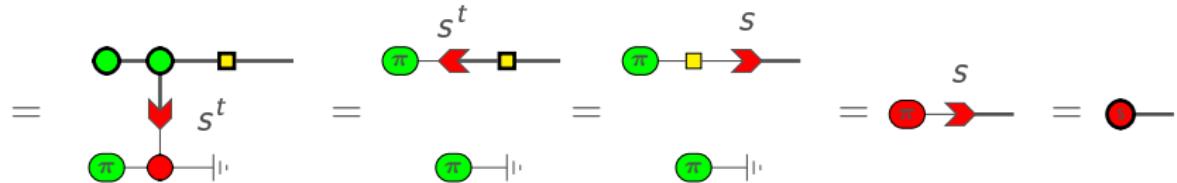
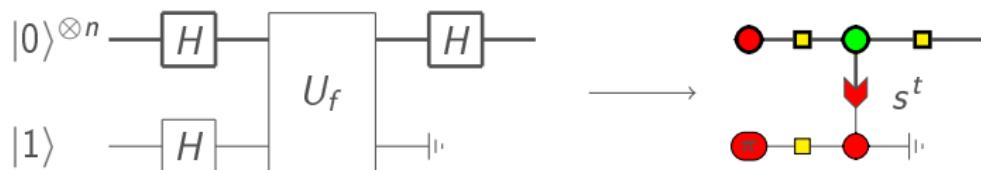
$$\begin{array}{c} A \\ \text{---} \end{array} \quad \begin{array}{c} B \\ \text{---} \end{array} = \begin{array}{c} \text{---} \end{array} A + B$$

Bernstein-Vazirani algorithm

Given $U_f : |x\rangle|y\rangle \mapsto |x\rangle|f(y)\rangle$ with:

$$f : \{0, 1\}^n \rightarrow \{0, 1\} \quad f : y \mapsto s^t \cdot y \quad s \in \{0, 1\}^n$$

we want to find s .



Chapter 5:

Stream ZX-Calculus

Stream types

- ⊕ ι is the type of one qubit at tick 1.
- ⊕ ιn is the type of n qubits at tick 1.
- ⊕ Given a gate $G : n \rightarrow m$ we can form $\iota G : \iota n \rightarrow \iota m$.
- ⊕ ω is the type of a stream of qubits.
- ⊕ Given a gate $G : n \rightarrow m$ we can form $\omega G : \omega n \rightarrow \omega m$.
- ⊕ $\triangleright^k \iota$ is the type of one qubit at tick k .
- ⊕ $\triangleright^k \omega$ is the type of a stream of qubits delayed by k ticks.
- ⊕ Given a diagram $D : x \rightarrow y$ we can form $\triangleright D : \triangleright x \rightarrow \triangleright y$.

Initialization and derivation

$$\omega \quad \text{---} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \quad \begin{matrix} \iota \\ \triangleright \omega \end{matrix}$$

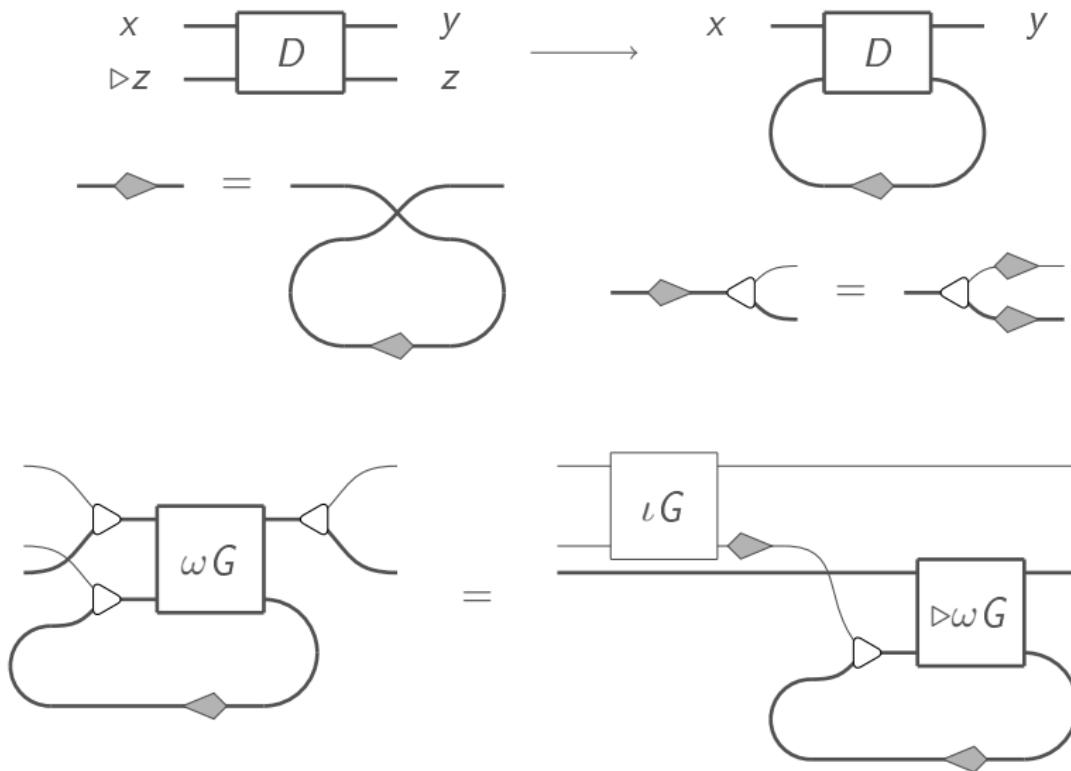
$$\begin{matrix} \iota \\ \triangleright \omega \end{matrix} \quad \text{---} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \quad \omega$$

$$\text{---} \quad = \quad \text{---}$$

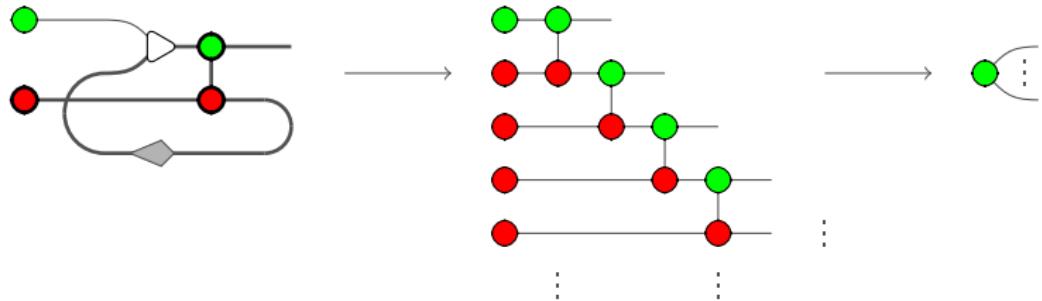
$$\text{---} \quad = \quad \text{---}$$

$$\begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array} \quad \boxed{\omega G} \quad \begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array} \quad = \quad \begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \quad \begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array} \quad \boxed{\iota G} \quad \begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \quad \begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array} \quad \boxed{\triangleright \omega G} \quad \begin{array}{c} \vdots \quad \vdots \\ \text{---} \quad \text{---} \end{array}$$

Delayed trace

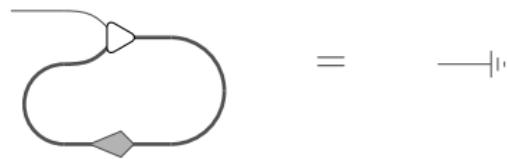


Cascades of CNOTs



Infinite storing

$$\frac{\Gamma, \triangleright S = \triangleright T \vdash S = T}{\Gamma \vdash S = T} \quad \triangleright$$



Future?

- ⌚ Iterating the construction we managed to simulate a quantum Turing machine. Work in progress.
- ⌚ Toward continuous variables and infinite dimensional quantum mechanics.
- ⌚ A unified theory of graphical languages (Props).
- ⌚ Graphical languages as a Rosetta stone between symbolic dynamic and condensed matter physics?